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Definability and textures ☆

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ABSTRACT

This paper aims to give a new perspective for definability in rough set theory. First, a counterpart of definability is introduced in textural approximation spaces. Then a complete field of sets for texture spaces is defined and using textural arguments, some new results are obtained for rough sets. It is shown that definability can be also discussed in terms of a complete field of fuzzy sets on a fuzzy lattice for the various fuzzy approximation spaces. It is also given a partial affirmative answer to an open problem posed by Wei-Zhi Wu in *On some mathematical structures of T-fuzzy rough set algebras in infinite universes of discourse* in Fundamenta Informaticae 108 (3–4) 2011 337–369.

0. Introduction

Definability is one of the primary concepts in rough set theory (see, e.g. [10–12,18,19]). Recall that a set is called definable if it is a union of some equivalence classes with respect to given equivalence relation [12]. This concept can be stated in terms of approximation operators. That is, a set is definable if the upper and lower approximations of it are equal. The pairs of rough set approximation operators and powersets form a category denoted by **R-APR** [7]. On the other hand, *a texture* is a family of sets satisfying certain conditions for a given universe. The basic motivation for textures is to provide a point-set based setting for fuzzy sets [1,2]. Duality is an essential phenomena in textures and then suitable morphisms are *direlations* between textures with two parts which are called *relation* and *corelation*, respectively. Complemented textures and complemented direlations form a category which is denoted by **cdrTex**, and **R-APR** is a full subcategory of **cdrTex**. Hence, the category **cdrTex** may be regarded as an abstract model for rough set theory (see [6,7]). In this paper, we do not follow the line containing categorical discussions. We introduce a counterpart of definability in **cdrTex** and in view of textural discussions, we present some new results in rough set theory. Recall that a complete field of sets on a universe is a family which is closed under arbitrary unions. Here, we consider a complete field of sets in texture spaces and then we show that such families can be stated using approximation operators. In [15], it is observed that direlations between Hutton textures turn into textural fuzzy direlations between fuzzy lattices (Hutton algebras). In [5], it is proved that if (ϕ, Φ) is a complemented textural fuzzy direlation on $\mathcal{F}(U)$, then the system $(\mathcal{F}(U), \wedge, \vee, \sim, \phi^{\leftarrow}, \Phi^{\leftarrow})$ defines a fuzzy rough set algebra where the lower and upper approximation operators $\phi^{\leftarrow}, \Phi^{\leftarrow} : \mathcal{F}(U) \to \mathcal{F}(U)$ are defined by

$$(\phi^{\leftarrow}\alpha)(u) = \bigvee\{s \in [0, 1] \mid \phi(u, s, v) \le \alpha(v)\ \forall v \in U\}$$

and

$$(\Phi^{\leftarrow}\alpha)(u) = \bigwedge\{s \in [0, 1] \mid \alpha(v) \le \Phi(u, s, v)\ \forall v \in U\}$$

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for any fuzzy set α, respectively. We use this fact and prove that definability can be also discussed in terms of a complete field of fuzzy sets on a fuzzy lattice for the various fuzzy rough set algebras. Clearly, every complete field of fuzzy sets is also a fuzzy σ-algebra. Then for the system $(\mathcal{F}(U), \wedge, \vee, \sim, \phi^{\leftarrow}, \Phi^{\leftarrow})$, we give a partial affirmative answer to an open problem related to fuzzy approximation spaces imposed by Wu in [20] (see Theorem 9.4). Furthermore, if the universe is finite, then the concepts of complete field and σ-algebra are coincide. Hence, for the finite case, our results are also true for σ-algebras.

This paper is an extension of our short conference paper [8]. Compared to [8], the present paper contains full proofs, more detailed remarks, and several further results.

For the benefit of the reader, we give the necessary concepts and results related to textures. The details on various concepts and results on textures given in Sections 1–6 may be found in [1–7,9,15].

1. Textures

Let U be a set. Then $\mathcal{U} \subseteq \mathcal{P}(U)$ is called a *texturing* of U, and $(U, \mathcal{U})$ is called a *texture space*, or simply a *texture*, if

(i) $(\mathcal{U}, \subseteq)$ is a complete lattice containing U and $\emptyset$, which has the property that arbitrary meets coincide with intersections, and finite joins coincide with unions,

(ii) $\mathcal{U}$ is completely distributive, that is, for all index set I, and for all $i \in I$, if J_i is an index set and if $A_i^j \in \mathcal{U}$, then we have

$$\bigcap_{i\in I} \bigvee_{j\in J_i} A_i^j = \bigvee_{\gamma\in\prod_i J_i} \bigcap A^i_{\gamma(i)},$$

(iii) $\mathcal{U}$ separates the points of U. That is, given $u_1 \neq u_2$ in U there exists $A \in \mathcal{U}$ such that $u_1 \in A,\ u_2 \notin A$, or $u_2 \in A,\ u_1 \notin A$.

A mapping $c_U : \mathcal{U} \to \mathcal{U}$ is called *a complementation* on $(U, \mathcal{U})$ if it satisfies the conditions $c_U^2(A) = A$ for all $A \in \mathcal{U}$ and $A \subseteq B$ in $\mathcal{U}$ implies $c_U(B) \subseteq c_U(A)$. Then the triple $(U, \mathcal{U}, c_U)$ is said to be *a complemented texture space*.

For $u \in U$, the p-sets and q-sets are defined by

$$P_u = \bigcap\{A \in \mathcal{U} \mid u \in A\} \quad \text{and} \quad Q_u = \bigvee\{A \in \mathcal{U} \mid u \notin A\}.$$

A nonempty set $A \in \mathcal{U}$ is *a molecule* if $\forall B, C \in \mathcal{U}, A \subseteq B \cup C \Rightarrow A \subseteq B$ or $A \subseteq C$. Clearly, p-sets are molecules of a texture space. A texture space $(U, \mathcal{U})$ is called *simple* if all molecules of the space are p-sets. The p-sets and the q-sets are important tools in the theory of texture spaces since the complete distributivity can be written in terms of p-sets and the q-sets.

Theorem 1.1 [4]. *Let $(\mathcal{U}, \subseteq)$ be a complete lattice. The following statements are equivalent.*

(i) *$(U, \mathcal{U})$ is completely distributive.*
(ii) *For $A, B \in \mathcal{U}$, if $A \not\subseteq B$ then there exists $u \in U$ with $A \not\subseteq Q_u$ and $P_u \not\subseteq B$.*

Example 1.2 [1]. (i) The pair $(U, \mathcal{P}(U))$ is a texture space where $\mathcal{P}(U)$ is the power set of U. It is called *a discrete texture*. Clearly, $(U, \mathcal{P}(U))$ is simple and for $u \in U$ we have

$$P_u = \{u\} \quad \text{and} \quad Q_u = U \setminus \{u\}$$

and $c_U : \mathcal{P}(U) \to \mathcal{P}(U)$ is the ordinary complementation on $(U, \mathcal{P}(U))$ defined by $c_U(A) = U \backslash A$ for all $A \in \mathcal{P}(U)$.

(ii) The family $\mathcal{M} = \{(0, r] \mid r \in [0, 1]\}$ is a texture on $M = (0, 1]$ which is called the Hutton texture. Clearly, $\mathcal{M}$ is closed under arbitrary intersections. Then it is easy to see that it is a complete lattice with respect to set inclusion. It is also completely distributive. To see this, take $(0, r], (0, s] \in \mathcal{M}$ where $(0, r] \not\subseteq (0, s]$. Then we have $s < r$. Choose a point $t \in [0, 1]$ where $s < t < r$. Since we have $P_t = Q_t = (0, t]$, we may write that $(0, r] \not\subseteq Q_t$ and $P_t \not\subseteq (0, s]$. Therefore, by Theorem 1.1. we obtain the complete distributivity of $\mathcal{M}$. Further, $\mathcal{M}$ is simple and the complementation $c_M : \mathcal{M} \to \mathcal{M}$ is defined by $\forall r \in (0, 1],\ c_M(0, r] = (0, 1 - r]$.

2. Products

Here, we discuss on the product of any two texture spaces $(U, \mathcal{U})$ and $(V, \mathcal{V})$. For the more information about the products of arbitrary families of textures can be found in [2]. Consider the family $\mathcal{A} = \{A \times V \mid A \in \mathcal{U}\} \bigcup \{U \times B \mid B \in \mathcal{V}\}$ and define

$$\mathcal{B} = \left\{ \bigcup_{j\in J} E_j \mid \{E_j\}_{j\in J} \subseteq \mathcal{A} \right\}.$$

The family of arbitrary intersections of the elements of $\mathcal{B}$, that is, the family

$$\mathcal{U} \otimes \mathcal{V} = \left\{ \bigcap_{i \in I} D_i \mid \{D_i\}_{i \in I} \subseteq \mathcal{B} \right\}$$

is a texture on $U \times V$. Clearly, for all $A \in \mathcal{U}$ and for all $B \in \mathcal{V}$, we have $A \times B \in \mathcal{U} \otimes \mathcal{V}$. Further, the p-sets and q-sets may be easily determined as

$$P_{(u,v)} = P_u \times P_v \quad \text{and} \quad Q_{(u,v)} = (U \times Q_v) \cup (Q_u \times V).$$

If c_U and c_V are complementations on the textures $(U, \mathcal{U})$ and $(V, \mathcal{V})$, respectively, then for the complementation $c_{U \times V}$ on the product, it is enough to check that

$$c_{U \times V}(U \times B) = U \times c_V(B) \quad \text{and} \quad c_{U \times V}(A \times V) = c_U(A) \times V$$

for all $A \in \mathcal{U}$ and $B \in \mathcal{V}$. In particular, if $\mathcal{P}(U)$ is a discrete texture on U, then for the textures $(U, \mathcal{P}(U))$, $(V, \mathcal{V})$, the p-sets and q-sets will be

$$\overline{P}_{(u,v)} = \{u\} \times P_v \text{ and } \overline{Q}_{(u,v)} = ((U \backslash \{u\}) \times V) \cup (U \times Q_v)$$

for the product texture $(U \times V, \ \mathcal{P}(U) \otimes \mathcal{V})$. Now take the texture $(M, \mathcal{M}, c_M)$ in Example 1.2 (ii). We clarify the product texture $\mathcal{P}(U) \otimes \mathcal{M}$ on $U \times M$. It is easy to see that the sets $A \times (0, r]$ are the elements of the product texture for all $A \subseteq U$ and $r \in [0, 1]$. Note that for $\mathcal{P}(U)$, we have $P_u = \{u\}$ and $Q_u = U \backslash \{u\}$ where $u \in U$. Further, we have $P_r = Q_r = (0, r] = Q_r$ in $\mathcal{M}$. Therefore, the p-sets and q-sets of the product texture $\mathcal{P}(U) \otimes \mathcal{M}$ are $P_{(u,r)} = P_u \times P_r = \{u\} \times (0, r]$ and $Q_{(u,r)} = (Q_u \times (0, 1]) \cup (U \times Q_r) = (U \backslash \{u\} \times (0, 1]) \cup (U \times (0, r])$, respectively. On the other hand, the complementations on $\mathcal{M}$ and $\mathcal{P}(U)$ are given by

$$\forall r \in (0, 1], \ c_{(0,1]}(0, r] = (0, 1 - r] \quad \text{and} \quad \forall A \subseteq U, \ c_U(A) = U \backslash A.$$

For the complementation $c_{U \times M}$ on the product texture $\mathcal{P}(U) \otimes \mathcal{M}$, we have

$$c_{U \times (0,1]}((A \times M) \cup (U \times (0, r])) = (U \backslash A) \times (0, 1 - r]$$

for every subset $A \subseteq U$ and $r \in M$.

3. Hutton textures

The basic motivation of textures is the correspondence between the fuzzy lattices and simple textures [2]. Let $(L, \leq, ')$ be a fuzzy lattice (Hutton algebra), that is, a complete, completely distributive lattice with an order reversing involution "$'$". Recall that $m \in L$ is *join-irreducible*, if

$$\forall a, b \in L, \quad m \leq a \vee b \Rightarrow m \leq a \quad \text{or} \quad m \leq b.$$

Consider the sets
$M_L = \{m \mid m \text{ is join-irreducible in } L\}$,
$\mathcal{M}_L = \{\widehat{a} \mid a \in L\}$, and
$\widehat{a} = \{m \mid m \in M_L \text{ and } m \leq a\}$, for all $a \in L$.

Then the mapping $\widehat{\ }: L \to \mathcal{M}_L$ defined by $\forall a \in L, \ a \mapsto \widehat{a}$ is a lattice isomorphism and the triple $(M_L, \mathcal{M}_L, c_{M_L})$ is a complemented simple texture space which is called *a Hutton texture*. Here the complementation $c_{M_L} : \mathcal{M}_L \to \mathcal{M}_L$ is defined by

$$\forall a \in L, \quad c_{M_L}(\widehat{a}) = \widehat{a'}.$$

Conversely, every complemented simple texture may be obtained in this way from a suitable Hutton algebra [2].

Example 3.1. (i) The unit interval $L = [0, 1]$ is a Hutton algebra with the usual ordering $\leq$ and the order reversing involution $'$ where $u' = 1 - u$ for all $u \in [0, 1]$. The corresponding simple texture to the Hutton algebra [0, 1] is the Hutton texture $(M, \mathcal{M}, c_M)$ given in Example 1.2 (ii) where

$$\mathcal{M}_L = \mathcal{M} = \{(0, u] \mid u \in [0, 1]\} \quad \text{and} \quad c_{M_L}(0, u] = c_M(0, u] = (0, 1 - u], \quad \forall u \in [0, 1].$$

Indeed, the set of all join-irreducible elements of [0, 1] is $M_L = (0, 1] = M$ and for every $u \in [0, 1]$, we have $\widehat{u} = (0, u]$. Then the mapping

$$\widehat{\ }: [0, 1] \longrightarrow \mathcal{M},$$
$$u \longrightarrow (0, u], \quad \forall u \in [0, 1]$$

is a lattice isomorphism.

(ii) Recall that a fuzzy subset α of U is a membership function $\alpha : U \to [0, 1]$. We denote the set of all fuzzy subsets of U by $\mathcal{F}(U)$. It is well known that $\mathcal{F}(U)$ is also an Hutton algebra with the pointwise ordering

$$\forall u \in U, \quad \alpha \leq \beta \Longleftrightarrow \alpha(u) \leq \beta(u)$$

and the order reversing involution $\alpha'(u) = 1 - \alpha(u)$. Here the join and the meet of fuzzy sets are considered as

$$(\alpha \wedge \beta)(u) = \alpha(u) \wedge \beta(u) \quad \text{and} \quad (\alpha \vee \beta)(u) = \alpha(u) \vee \beta(u)$$

for all $\alpha, \beta \in \mathcal{F}(U)$.

Now consider the fuzzy points u_s and fuzzy copoints u^s of $\mathcal{F}(U)$ defined by

$$u_s(z) = \begin{cases} s, & \text{if } z = u, \\ 0, & \text{if } z \neq u, \end{cases} \quad \text{and} \quad u^s(z) = \begin{cases} s, & \text{if } z = u, \\ 1, & \text{if } z \neq u, \end{cases}$$

for all $z \in U$, respectively [2,15]. Let us take the sets:

$\widehat{\alpha} = \{u_s \mid u_s \leq \alpha\}$,

$\mathcal{M}_{\mathcal{F}(U)} = \{\widehat{\alpha} \mid \alpha \in \mathcal{F}(U)\}$, and

$M_{\mathcal{F}(U)} = \{u_s \mid u_s \text{ is a fuzzy point in } \mathcal{F}(U)\ \}$.

Then under the lattice isomorphim $\widehat{\ }: \mathcal{F}(U) \to \mathcal{M}_{\mathcal{F}(U)}$, the corresponding texture space will be $(M_{\mathcal{F}(U)}, \mathcal{M}_{\mathcal{F}(U)})$. Every fuzzy point u_s can be regarded as an ordered pair $(u, s) \in U \times (0, 1]$ and then we may write that $\widehat{\alpha} = \{(u, s) \mid s \leq \alpha(u)\}$. Therefore, it can be shown that the texture $(M_{\mathcal{F}(U)}, \mathcal{M}_{\mathcal{F}(U)})$ is isomorphic to the product texture

$(U \times M,\ \mathcal{P}(U) \otimes \mathcal{M},\ c_{U\times M})$

of $(U, \mathcal{P}(U), c_U)$ and $(M, \mathcal{M}, c_M)$ while the complementation mapping is defined by

$c_{U\times M}\,(\widehat{\alpha}) = \widehat{1 - \alpha}$

for all $\alpha \in \mathcal{F}(U)$ [2]. Meanwhile, we immediately have that

$\widehat{u_s} = \{u\} \times (0, s] = P_{(u,s)}$ and $\widehat{u^s} = (U\backslash\{u\} \times [0, 1]) \cup (U \times (0, s]) = Q_{(u,s)}$.

4. Direlations

Duality is an essential phenomena in textures and then suitable morphisms are direlations between textures with two parts which are called relation and corelation, respectively [3]. Now let $(U, \mathcal{U})$, $(V, \mathcal{V})$ be texture spaces and let us consider the product texture $\mathcal{P}(U) \otimes \mathcal{V}$ of the texture spaces $(U, \mathcal{P}(U))$ and $(V, \mathcal{V})$ and denote the p-sets and the q-sets by $\overline{P}_{(u,v)}$ and $\overline{Q}_{(u,v)}$ respectively. Then

(i) $r \in \mathcal{P}(U) \otimes \mathcal{V}$ is called a *relation* from $(U, \mathcal{U})$ to $(V, \mathcal{V})$ if it satisfies

R1 $r \not\subseteq \overline{Q}_{(u,v)}, P_{u'} \not\subseteq Q_u \implies r \not\subseteq \overline{Q}_{(u',v)}$.

R2 $r \not\subseteq \overline{Q}_{(u,v)} \implies \exists u' \in U$ such that $P_u \not\subseteq Q_{u'}$ and $r \not\subseteq \overline{Q}_{(u',v)}$.

(ii) $R \in \mathcal{P}(U) \otimes \mathcal{V}$ is called a *corelation* from $(U, \mathcal{U})$ to $(V, \mathcal{V})$ if it satisfies

CR1 $\overline{P}_{(u,v)} \not\subseteq R, P_u \not\subseteq Q_{u'} \implies \overline{P}_{(u',v)} \not\subseteq R$.

CR2 $\overline{P}_{(u,v)} \not\subseteq R \implies \exists u' \in U$ such that $P_{u'} \not\subseteq Q_u$ and $\overline{P}_{(u',v)} \not\subseteq R$.

A pair (r, R), where r is a relation and R a corelation from $(U, \mathcal{U})$ to $(V, \mathcal{V})$ is called a *direlation from* $(U, \mathcal{U})$ *to* $(V, \mathcal{V})$.

Note that if (r, R) is a direlation from the texture $(U, \mathcal{P}(U))$ to $(V, \mathcal{P}(V))$, then r and R are point relations from U to V, that is, $r, R \subseteq U \times V$ since $\mathcal{P}(U) \otimes \mathcal{P}(V) = \mathcal{P}(U \times V)$. The identity direlation (i, I) on $(U, \mathcal{U})$ is defined by

$$i = \bigvee\{\overline{P}_{(u,u)} \mid u \in U\} \quad \text{and} \quad I = \bigcap\{\overline{Q}_{(u,u)} \mid u \in U^\flat\}$$

where $U^\flat = \{u \mid U \not\subseteq Q_u\}$. Recall that if (r, R) is a direlation on $(U, \mathcal{U})$, then r is *reflexive* if $i \subseteq r$ and R is *reflexive* if $R \subseteq I$. Then we say that (r, R) is *reflexive* if r and R are reflexive.

Now let (r, R) be a direlation from $(U, \mathcal{U})$ to $(V, \mathcal{V})$ where $(U, \mathcal{U})$ and $(V, \mathcal{V})$ are any two texture spaces. Then *the inverses* of r and R are defined by

$$r^{\leftarrow} = \bigcap \left\{\overline{Q}_{(v,u)} \mid r \not\subseteq \overline{Q}_{(u,v)}\right\} \quad \text{and} \quad R^{\leftarrow} = \bigvee \left\{\overline{P}_{(v,u)} \mid \overline{P}_{(u,v)} \not\subseteq R\right\},$$

respectively where $r^{\leftarrow}$ is a corelation and $R^{\leftarrow}$ is a relation.

Further, the direlation $(r, R)^{\leftarrow} = (R^{\leftarrow}, r^{\leftarrow})$ from $(V, \mathcal{V})$ to $(U, \mathcal{U})$ is called the *inverse* of the direlation (r, R). Then (r, R) is called *symmetric* if $r = R^{\leftarrow}$ and $R = r^{\leftarrow}$.

The A-sections and the B-presections with respect to relation and corelation are given as

$$r^{\rightarrow}A = \bigcap \left\{Q_v \mid \forall u, r \not\subseteq \overline{Q}_{(u,v)} \Rightarrow A \subseteq Q_u\right\}$$

$$R^{\rightarrow}A = \bigvee \{P_v \mid \forall u, \overline{P}_{(u,v)} \not\subseteq R \Rightarrow P_u \subseteq A\}$$

$$r^{\leftarrow}B = \bigvee \left\{P_u \mid \forall v, r \not\subseteq \overline{Q}_{(u,v)} \Rightarrow P_v \subseteq B\right\}, \quad \text{and}$$

$$R^{\leftarrow}B = \bigcap \{Q_u \mid \forall v, \overline{P}_{(u,v)} \not\subseteq R \Rightarrow B \subseteq Q_v\}$$

for all $A \in \mathcal{U}$ and $B \in \mathcal{V}$, respectively.

Now let $(U, \mathcal{U})$, $(V, \mathcal{V})$, $(W, \mathcal{W})$ be texture spaces. For any relation p from $(U, \mathcal{U})$ to $(V, \mathcal{V})$ and for any relation q from $(V, \mathcal{V})$ to $(W, \mathcal{W})$ their *composition* $q \circ p$ from $(U, \mathcal{U})$ to $(W, \mathcal{W})$ is defined by

$$q \circ p = \bigvee \left\{\overline{P}_{(u,w)} \mid \exists\, v \in V \text{ with } p \not\subseteq \overline{Q}_{(u,v)} \text{ and } q \not\subseteq \overline{Q}_{(v,w)}\right\}$$

and any corelation P from $(U, \mathcal{U})$ to $(V, \mathcal{V})$ and for any corelation Q from $(U, \mathcal{U})$ to $(V, \mathcal{V})$ their *composition* $Q \circ P$ from $(U, \mathcal{U})$ to $(W, \mathcal{W})$ defined by

$$Q \circ P = \bigcap \left\{\overline{Q}_{(u,w)} \mid \exists\, v \in V \text{ with } \overline{P}_{(u,v)} \not\subseteq P \text{ and } \overline{P}_{(v,w)} \not\subseteq Q\right\}.$$

Finally, the *composition* of the direlations (p, P), (q, Q) is the direlation

$$(q, Q) \circ (p, P) = (q \circ p, Q \circ P).$$

Further, r is *transitive* if $r \circ r \subseteq r$ and R is *transitive* if $R \subseteq R \circ R$. Then we say that (r, R) is *transitive* if r and R are transitive.

Now let c_U and c_V be the complementations on $(U, \mathcal{U})$ and $(V, \mathcal{V})$, respectively. The complement r' of the relation r is the corelation

$$r' = \bigcap \left\{\overline{Q}_{(u,v)} \mid \exists w, z \text{ with } r \not\subseteq \overline{Q}_{(w,z)}, c_U(Q_u) \not\subseteq Q_w \text{ and } P_z \not\subseteq c_V(P_v)\right\}.$$

The complement R' of the corelation R is the relation

$$R' = \bigvee \{\overline{P}_{(u,v)} \mid \exists w, z \text{ with } \overline{P}_{(w,z)} \not\subseteq R, P_w \not\subseteq c_U(P_u) \text{ and } c_V(Q_v) \not\subseteq Q_z\}.$$

The complement $(r, R)'$ of the direlation (r, R) is the direlation $(r, R)' = (R', r')$. A direlation (r, R) is called *complemented* if $r = R'$ and $R = r'$.

5. Textural rough set algebras

Let (r, R) be a direlation on a texture $(U, \mathcal{U})$. Then the quadruple $(U, \mathcal{U}, r, R)$ is called *a textural approximation space*. If (r, R) is a complemented direlation, then we say $(U, \mathcal{U}, r, R)$ is *a complemented textural approximation space*. Presections satisfy significant properties as rough sets [3,6]. In this section, we recall some basic results on presections. Some of them are already proved in [3].

Lemma 5.1. *For all $A, B \in \mathcal{U}$ and the family $\{A_j \mid j \in J\} \subseteq \mathcal{U}$, presections satisfy the following properties:*

(a) $A \subseteq B \implies r^{\leftarrow}A \subseteq r^{\leftarrow}B$.

(b) $A \subseteq B \implies R^{\leftarrow}A \subseteq R^{\leftarrow}B$.

(c) $\bigvee_{j\in J} r^{\leftarrow}A_j \subseteq r^{\leftarrow} \bigvee_{j\in J} A_j$.

(d) $r^{\leftarrow} \bigcap_{j\in J} A_j = \bigcap_{j\in J} r^{\leftarrow}A_j$.

(e) $\bigvee_{j\in J} R^{\leftarrow}A_j = R^{\leftarrow} \bigvee_{j\in J} A_j$.

(f) $R^{\leftarrow} \bigcap_{j\in J} A_j \subseteq \bigcap_{j\in J} R^{\leftarrow}A_j$.

(g) $r^{\leftarrow}U = U$ *and* $R^{\leftarrow}\emptyset = \emptyset$.

Theorem 5.2. *If (r, R) is a complemented direlation on the complemented texture space $(U, \mathcal{U}, c_U)$, then*

$$c_U r^{\leftarrow} A = R^{\leftarrow} c_U A \quad \text{and} \quad c_U R^{\leftarrow} A = r^{\leftarrow} c_U A.$$

Let $\mathbf{L}, \mathbf{H} : \mathcal{U} \to \mathcal{U}$ be two unary operators. Then $\mathbf{L}$ and $\mathbf{H}$ are called *dual operators on* $(U, \mathcal{U})$ if,

$$c_U \mathbf{L}(A) = \mathbf{H}(c_U(A)) \quad \text{and} \quad c_U \mathbf{H}(A) = \mathbf{L}(c_U(A))$$

for all $A \in \mathcal{U}$. Now consider the following conditions:

($\mathbf{L}_1$) $L(U) = U$, ($\mathbf{H}_1$) $H(\emptyset) = \emptyset$,

($\mathbf{L}_2$) $L(\bigcap_{j \in J} A_j) = \bigcap_{j \in J} L(A_j)$. ($\mathbf{H}_2$) $H(\bigvee_{j \in J} A_j) = \bigvee_{j \in J} H(A_j)$.

If two dual operators L and H satisfy the conditions $\mathbf{L}_1$ and $\mathbf{L}_2$ or equivalently, $\mathbf{H}_1$ and $\mathbf{H}_2$, then the system $(\mathcal{U}, \vee, \cap, c_U, L, H)$ defines a (textural) rough set algebra in the sense of Yao [17], and the operators L and H are called *approximation operators* on $(U, \mathcal{U})$.

By Lemma 6.4 and Theorem 5.2, the system $(\mathcal{U}, \cap, \vee, c_U, r^{\leftarrow}, R^{\leftarrow})$ is a textural rough set algebra where

$$r^{\leftarrow}, R^{\leftarrow} : \mathcal{U} \to \mathcal{U}$$

are approximation operators defined by

$$r^{\leftarrow} A = \bigvee \left\{ P_u \mid \forall v, r \not\subseteq \overline{Q}_{(u,v)} \Rightarrow P_v \subseteq A \right\}, \quad \text{and}$$

$$R^{\leftarrow} A = \bigcap \{ Q_u \mid \forall v, \overline{P}_{(u,v)} \not\subseteq R \Rightarrow A \subseteq Q_v \}$$

for all $A \in \mathcal{U}$. Then the pair $(r^{\leftarrow} A, R^{\leftarrow} A)$ is called a textural rough set.

The following results give an idea for the axiomatic structure of textural rough sets.

Theorem 5.3. *Let $L, H : \mathcal{U} \to \mathcal{U}$ be dual operators on the complemented texture space $(U, \mathcal{U}, c_U)$. Then there exists a unique complemented direlation (r, R) on $(U, \mathcal{U})$ such that*

$$L(A) = r^{\leftarrow} A \text{ and } H(A) = R^{\leftarrow} A$$

for all $A \in \mathcal{U}$ if and only if L and H satisfy the equivalent properties

($\mathbf{L}_1$) $L(U) = U$,

($\mathbf{L}_2$) $L(\bigcap_{j \in J} A_j) = \bigcap_{j \in J} L(A_j)$, *and*

($\mathbf{H}_1$) $H(\emptyset) = \emptyset$,

($\mathbf{H}_2$) $H(\bigvee_{j \in J} A_j) = \bigvee_{j \in J} H(A_j)$.

Theorem 5.4. *Let $L, H : \mathcal{U} \to \mathcal{U}$ be dual operators. If L satisfies $\mathbf{L}_1$, $\mathbf{L}_2$ and the axioms*

$\mathbf{L}_3$ $L(A) \subseteq A$,

$\mathbf{L}_4$ $L(L(A)) = L(A)$, *and*

$\mathbf{L}_5$ $c_U(L(c_U(L(A)) \subseteq A$,

then there exists a unique complemented equivalence direlation (r, R) on $(U, \mathcal{U})$ such that

$$L(A) = r^{\leftarrow}(A) \quad \text{and} \quad H(A) = R^{\leftarrow}(A)$$

for all $A \in \mathcal{U}$.

Recall that if r is a point relation on U, that is, $r \subseteq U \times U$, then the generalized rough set based on the point relation r is given by $(\underline{apr}_r A, \ \overline{apr}_r A)$ where

$\underline{apr}_r A = \{x \mid \forall y \in U, \ (x, y) \in r \implies y \in A\}$, and

$\overline{apr}_r A = \{x \mid \exists y \in U, \ (x, y) \in r \text{ and } y \in A\}$

for all $A \subseteq U$ (see, e.g. [16]). On the other hand, the pair $(r, \ (U \times U) \setminus r)$ can be regarded as a complemented direlation on the discrete texture $(U, \mathcal{P}(U))$ where $R = (U \times U) \setminus r$. Conversely, if (r, R) is a complemented direlation on $(U, \mathcal{P}(U))$, then r and R are point relations on U where $R = (U \times U) \setminus r$. Therefore, using the facts

(1) $r \not\subseteq \overline{Q}_{(u,v)} \iff (u, v) \in r$, and

(2) $\overline{P}_{(u,v)} \not\subseteq R \iff (u, v) \notin R$,

we immediately conclude that

$$(r^{\leftarrow}A,\ R^{\leftarrow}A) = \left(\underline{apr}_r A,\ \overline{apr}_r A\right)$$

for every set $A \in \mathcal{P}(U)$. Now we have.

Theorem 5.5. *If r is a point relation on U, that is, $r \subseteq U \times U$, then*

$$\forall X \subseteq U, \quad \underline{apr}_r X = U \backslash r^{-1}(U \backslash X) = r^{\leftarrow}X \quad \textit{and} \quad \overline{apr}_r X = r^{-1}(X) = R^{\leftarrow}X.$$

6. Textural fuzzy direlations

Textural fuzzy direlations between any two fuzzy lattices are introduced in [15]. Let us denote the texture space $(U \times (0, 1],\ \mathcal{P}(U) \otimes \mathcal{M})$ by $(W_U, \mathcal{W}_U)$ where $\mathcal{M}$ is the texturing in Example 3.1 (i). Consider the fuzzy lattice $\mathcal{F}(U \times [0, 1] \times U)$, that is, the family of all fuzzy subsets

$$\phi : U \times [0, 1] \times U \to [0, 1]$$

of the set $U \times [0, 1] \times U$. Clearly, the corresponding texture is $\mathcal{P}(U \times (0, 1] \times U) \otimes \mathcal{M}$. It is easy to see that the textures

$$\mathcal{P}(U \times (0, 1] \times U) \otimes \mathcal{M} \ \text{ and } \ \mathcal{P}(U \times (0, 1]) \otimes (\mathcal{P}(U) \otimes \mathcal{M})$$

are isomorphic where $\mathcal{P}(U \times (0, 1]) \otimes (\mathcal{P}(U) \otimes \mathcal{M}) = \mathcal{P}(W_U) \otimes \mathcal{W}_U$. Hence, if we consider the lattice isomorphism $\widehat{\ }: \mathcal{F}(U \times [0, 1] \times U) \to \mathcal{P}(W_U) \otimes \mathcal{W}_U$, then for all $\phi \in \mathcal{F}(U \times [0, 1] \times U)$, we obtain

$$\widehat{\phi} = \{((u, s), (v, t)) \mid t \leq \phi(u, s, v)\}.$$

Hence, if r is a relation or a corelation on the texture $(W_U, \mathcal{W}_U, w_U)$, then we have $r \in \mathcal{P}(W_U) \otimes \mathcal{W}_U$ and so for some $\mu_r : U \times [0, 1] \times U \to [0, 1]$, we may write that

$$\widehat{\mu_r} = r \ \text{ and } \ \widehat{\mu_r} = \{((u, s), (v, t)) \mid t \leq \mu_r(u, s, v)\}.$$

Now let us consider the following definition.

Definition 6.1. Let $\phi, \Phi \in \mathcal{F}(U \times [0, 1] \times U)$.

(1) ϕ is called *a textural fuzzy relation* on $\mathcal{F}(U)$ if

$$\phi(u, s, v) = \bigvee\{\phi(u, s', v) \mid 0 < s' < s\}, \forall (u, s, v) \in U \times [0, 1] \times U.$$

(2) Φ is called *a textural fuzzy corelation* on $\mathcal{F}(U)$ if

$$\Phi(u, s, v) = \bigwedge\{\Phi(u, s', v) \mid s < s' \leq 1\}, \forall (u, s, v) \in U \times [0, 1] \times U.$$

(3) If ϕ is a textural fuzzy relation and Φ is a textural fuzzy corelation, (ϕ, Φ) is called *a textural fuzzy direlation* on $\mathcal{F}(U)$.

Definition 6.2. Let (ϕ, Φ) be a textural fuzzy direlation on $\mathcal{F}(U)$. Then

(i) $\beta \in \mathcal{F}(U)$ is called *the α-section* of the textural fuzzy relation ϕ on $\mathcal{F}(U)$ if

$$\beta(v) = \bigwedge\{t \in [0, 1] \mid s < \alpha(u) \implies \phi(u, s, v) \leq t\}, \quad \forall v \in U.$$

(ii) $\beta \in \mathcal{F}(U)$ is called *the α-section* of the textural fuzzy corelation Φ on $\mathcal{F}(U)$ if

$$\beta(v) = \bigvee\{t \in [0, 1] \mid \alpha(u) < s \implies t \leq \Phi(u, s, v)\}, \quad \forall v \in U.$$

If β is the α-section of ϕ, it is denoted by $\phi^{\rightarrow}\alpha = \beta$. Similarly, if β is the α-section of Φ, it is denoted by $\Phi^{\rightarrow}\alpha = \beta$.

Definition 6.3. Let (ϕ, Φ) be a textural fuzzy direlation on $\mathcal{F}(U)$. Then the pair $(\Phi^{\leftarrow}, \phi^{\leftarrow})$ is called *the inverse direlation* of (ϕ, Φ) where

$$\phi^{\leftarrow}(v, t, u) = \bigvee\{s \in [0, 1] \mid \phi(u, s, v) \leq t\}, \quad \text{and}$$

$$\Phi^{\leftarrow}(v, t, u) = \bigwedge\{s \in [0, 1] \mid t \leq \Phi(u, s, v)\}$$

for all $(v, t, u) \in U \times [0, 1] \times U$.

Lemma 6.4. *Let (ϕ, Φ) be a textural fuzzy direlation on $\mathcal{F}(U)$ and $\alpha \in \mathcal{F}(U)$. Then*

(i) $(\phi^{\leftarrow}\alpha)(u) = \bigvee\{s \in [0,1] \mid \phi(u,s,v) \leq \alpha(v)\ \forall v \in U\}$.

(ii) $(\Phi^{\leftarrow}\alpha)(u) = \bigwedge\{s \in [0,1] \mid \alpha(v) \leq \Phi(u,s,v)\ \forall v \in U\}$.

Note that if (ϕ, Φ) is a textural fuzzy direlation on $\mathcal{F}(U)$, then by Theorem 2.3 in [15] there exists a direlation (r, R) on $(W_U, \mathcal{W}_U)$ such that $\phi = \mu_r$ and $\Phi = \mu_R$. Therefore, we may write that

$$\phi^{\leftarrow}(v,t,u) = \bigvee\{s \in [0,1] \mid \mu_r(u,s,v) \leq t\} = \mu_{r^{\leftarrow}}(v,t,u), \quad \text{and}$$

$$\Phi^{\leftarrow}(v,t,u) = \bigwedge\{s \in [0,1] \mid t \leq \mu_R(u,s,v)\} = \mu_{R^{\leftarrow}}(v,t,u)$$

and since $\phi^{\leftarrow} = \mu_r^{\leftarrow}$ and $\Phi^{\leftarrow} = \mu_R^{\leftarrow}$, we have

$$\mu_{r^{\leftarrow}} = \mu_r^{\leftarrow} \quad \text{and} \quad \mu_{R^{\leftarrow}} = \mu_R^{\leftarrow}.$$

On the other hand, we may write the following equivalences.

$$\mu_r^{\leftarrow}\alpha = \beta \Longleftrightarrow r^{\leftarrow}\widehat{\alpha} = \widehat{\beta}, \quad \mu_R^{\leftarrow}\alpha = \beta \Longleftrightarrow R^{\leftarrow}\widehat{\alpha} = \widehat{\beta}$$

and

$$\mu_r^{\rightarrow}\alpha = \beta \Longleftrightarrow r^{\rightarrow}\widehat{\alpha} = \widehat{\beta}, \quad \mu_R^{\rightarrow}\alpha = \beta \Longleftrightarrow R^{\rightarrow}\widehat{\alpha} = \widehat{\beta}.$$

If (i, I) is the identity direlation on $(W_U, \mathcal{W}_U)$, then the corresponding fuzzy textural direlation (μ_i, μ_I) on $\mathcal{F}(U)$ is given by

$$\mu_i(u,s,v) = \begin{cases} 0, & \text{if } u \neq v, \\ s, & \text{if } u = v, \end{cases} \qquad \mu_I(u,s,v) = \begin{cases} 1, & \text{if } u \neq v, \\ s, & \text{if } u = v, \end{cases}$$

The pair (μ_i, μ_I) is called *the textural fuzzy identity direlation* on $\mathcal{F}(U)$.

Finally, we recall the following definitions:

(i) (ϕ, Φ) is called *reflexive* if $\mu_i \leq \phi$ and $\Phi \leq \mu_I$.

(ii) (ϕ, Φ) is called *symmetric* if $\phi^{\leftarrow} = \Phi$ and $\Phi^{\leftarrow} = \phi$.

(iii) (ϕ, Φ) is called *transitive* if $\phi \circ \phi \leq \phi$ and $\Phi \leq \Phi \circ \Phi$.

(iv) (ϕ, Φ) is called *an equivalence fuzzy direlation* if it is reflexive, symmetric and transitive.

7. Textural definability

As it is mentioned in Section 4, the system $(\mathcal{U}, \cap, \vee, c_U, r^{\leftarrow}, R^{\leftarrow})$ is a textural rough set algebra where $r^{\leftarrow}, R^{\leftarrow} : \mathcal{U} \to \mathcal{U}$ are the approximation operators. Definability can be given in terms of presections.

Definition 7.1. Let (r, R) be a complemented direlation on $(U, \mathcal{U})$. Then the set $A \in \mathcal{U}$ is called *definable* if $r^{\leftarrow}A = R^{\leftarrow}A$.

Difunctions are important tools for textures as morphisms of the category **dfTex** whose objects are textures [3]. *A difunction* on a texture $(U, \mathcal{U})$ is a direlation (r, R) satisfying the following two conditions:

DF1 For $u, v \in U$, $P_u \not\subseteq Q_v \implies \exists\, w \in U$ with $r \not\subseteq \overline{Q}_{(u,w)}$ and $\overline{P}_{(v,w)} \not\subseteq R$.

DF2 For $u, v \in U$ and $w \in U$, $r \not\subseteq \overline{Q}_{(w,u)}$ and $\overline{P}_{(w,v)} \not\subseteq R \implies P_v \not\subseteq Q_u$.

Let us note that there is a close relation between difunctions and definability, i.e., textural definability characterizes difunctions.

Theorem 7.2 [3]. *Let (r, R) be a direlation on $(U, \mathcal{U})$. Then the following conditions are equivalent:*

(i) *(r, R) is a difunction on $(U, \mathcal{U})$.*

(ii) *Every set A in $\mathcal{U}$ is definable that is, $r^{\leftarrow}A = R^{\leftarrow}A$.*

Theorem 7.3. *Let (r, R) be a reflexive and symmetric direlation on a texture $(U, \mathcal{U})$ and $A \in \mathcal{U}$. Then the following conditions are equivalent.*

(i) *A is definable.*

(ii) $r^{\leftarrow}A = A$.

(iii) $R^{\leftarrow}A = A$.

Proof. (i) $\Longrightarrow$ (ii) Let A be definable, that is, let $r^{\leftarrow}A = R^{\leftarrow}A$. Since (r, R) is reflexive, by Theorem 4.2 in [5], we have $r^{\leftarrow}A \subseteq A$ and $A \subseteq R^{\leftarrow}A$. This implies that $r^{\leftarrow}A = A$.

(ii) $\Longrightarrow$ (iii) Since (r, R) is symmetric, by Theorem 4.8 in [6], we may write that $R^{\leftarrow}r^{\leftarrow}A \subseteq A$ and so by the assumption we find $R^{\leftarrow}A \subseteq A$. Since R is reflexive, we find $R^{\leftarrow}A = A$.

(iii) $\Longrightarrow$ (i) Suppose that $R^{\leftarrow}A = A$. Since r is reflexive, $r^{\leftarrow}A \subseteq A$ and hence, $r^{\leftarrow}A \subseteq R^{\leftarrow}A$. Further, by Theorem 4.8 in [6], $A \subseteq r^{\leftarrow}R^{\leftarrow}A$ and this implies that $A \subseteq r^{\leftarrow}A$, that is $R^{\leftarrow}A \subseteq r^{\leftarrow}A$. □

Theorem 7.4. *If (r, R) is an equivalence direlation, then for all $A \in \mathcal{U}$ the presections $r^{\leftarrow}A$ and $R^{\leftarrow}A$ are definable.*

Proof. Since (r, R) is reflexive and transitive, by Theorem 4.7 in [6], we have $r^{\leftarrow}r^{\leftarrow}A = r^{\leftarrow}A$ and $R^{\leftarrow}R^{\leftarrow}A = R^{\leftarrow}A$. Then by Theorem 7.3, we obtain that $r^{\leftarrow}A$ and $R^{\leftarrow}A$ are definable. □

The following is a textural counterpart of a family of complete field of sets.

Definition 7.5. Let $(U, \mathcal{U}, c_U)$ be a complemented texture space and $\mathcal{D} \subseteq \mathcal{U}$. Then $\mathcal{D}$ is called *a textural complete field of sets* on $(U, \mathcal{U}, c_U)$ if the following conditions hold:

(i) $U \in \mathcal{D}$,

(ii) $\mathcal{G} \subseteq \mathcal{D} \implies \bigvee \mathcal{G} \in \mathcal{D}$,

(iii) $G \in \mathcal{D} \implies c_U(G) \in \mathcal{D}$.

In the above definition, clearly, we have $\emptyset \in \mathcal{D}$. Further, if $\mathcal{G} \subseteq \mathcal{D}$, then we also have

$$\bigcap \mathcal{G} = c_U\left(\bigvee\{c_U(G) \mid G \in \mathcal{G}\}\right) \in \mathcal{D}.$$

Theorem 7.6. *Let (r, R) be a complemented direlation on $(U, \mathcal{U})$ and $\mathcal{D}$ be the family of all definable sets, that is, $\mathcal{D} = \{A \in \mathcal{U} \mid r^{\leftarrow}A = R^{\leftarrow}A\}$. Then we have the following.*

(i) *If (r, R) is reflexive, then $\mathcal{D}$ is a textural complete field of sets.*

(ii) *If (r, R) is an equivalence direlation, then*

$$\forall A \in \mathcal{U},\ r^{\leftarrow}A = \bigvee\{B \in \mathcal{D} \mid B \subseteq A\} \quad \textit{and} \quad R^{\leftarrow}A = \bigcap\{B \in \mathcal{D} \mid A \subseteq B\}.$$

Proof. (i) By Lemma 4.1 (g) in [6], we have $r^{\leftarrow}U = U$. Since R is reflexive, by Theorem 4.4. (ii) in [6], $U \subseteq R^{\leftarrow}U$, that is $R^{\leftarrow}U = U$ and so we find that $U \in \mathcal{D}$. Now let $A \in \mathcal{D}$. In this case,

$$r^{\leftarrow}c_U(A) = c_U(R^{\leftarrow}A) = c_U(r^{\leftarrow}A) = R^{\leftarrow}c_U(A).$$

This follows that $c_U(A) \in \mathcal{D}$. Let $\{A_j \mid j \in J\} \subseteq \mathcal{D}$. We show that $R^{\leftarrow}\bigvee_{j\in J} A_j = r^{\leftarrow}\bigvee_{n\in J} A_j$. Since (r, R) is reflexive, by Theorem 4.4 in [6], we may write that

$$r^{\leftarrow}\bigvee_{j\in J} A_j \subseteq \bigvee_{j\in J} A_j \subseteq R^{\leftarrow}\bigvee_{j\in J} A_j.$$

By Corollary 2.12 in [3], we have

$$R^{\leftarrow}\bigvee_{j\in J} A_j = \bigvee_{j\in J} R^{\leftarrow}A_j = \bigvee_{j\in J} r^{\leftarrow}A_j \subseteq r^{\leftarrow}\bigvee_{j\in J} A_j$$

and this gives us the desired equality.

(ii) Since r is reflexive, we have $r^{\leftarrow}A \subseteq A$. By Theorem 7.4, $r^{\leftarrow}A$ is definable and so $r^{\leftarrow}A \in \mathcal{D}$. Hence we have $r^{\leftarrow}A \subseteq \bigvee\{B \in \mathcal{D} | B \subseteq A\}$. For the reverse inclusion, let $B \subseteq A$ and $B \in \mathcal{D}$. Then we may write that $r^{\leftarrow}B \subseteq r^{\leftarrow}A$. Further, by Theorem 7.3, $r^{\leftarrow}B = B$ and this completes the proof. □

Theorem 7.7. *Let $\mathcal{D}$ be a textural complete field of sets on a complemented texture $(U, \mathcal{U}, c_U)$. Then there exists a complemented equivalence direlation (r, R) on $(U, \mathcal{U})$ such that*

$$\mathcal{D} = \{A \in \mathcal{U} \mid r^{\leftarrow}A = R^{\leftarrow}A\}.$$

Proof. Let us consider the operators $L, H : \mathcal{U} \longrightarrow \mathcal{U}$ defined by

$L(A) = \bigvee\{B \mid B \subseteq A \text{ and } B \in \mathcal{D}\}$

and

$H(A) = \bigcap\{C \mid A \subseteq C \text{ and } C \in \mathcal{D}\}$

for all $A \in \mathcal{U}$, respectively. Since we also have $\mathcal{D} = \{c_U(A) \mid A \in \mathcal{D}\}$, it is easy to see that L and H are dual operators. Let us show that L satisfies the following conditions:

($\mathbf{L}_1$) $L(U) = U$,

($\mathbf{L}_2$) $L(\bigcap_{j\in J} A_j) = \bigcap_{j\in J} L(A_j)$,

($\mathbf{L}_3$) $L(A) \subseteq A$,

($\mathbf{L}_4$) $L(L(A)) = L(A)$,

($\mathbf{L}_5$) $c_U(L(c_U(L(A)))) \subseteq A$.

First, we verify the conditions ($\mathbf{L}_1$), ($\mathbf{L}_3$) and ($\mathbf{L}_4$). Since $U \in \mathcal{D}$, clearly $L(U) = U$. By definition of L we immediately have $L(A) \subseteq A$. Furthermore, since $L(A) \in \mathcal{D}$, $L(L(A)) = L(A)$. The mapping L is monotonic. Indeed, take $A, B \in \mathcal{U}$ where $A \subseteq B$. This implies that $L(A) \subseteq B$ and since $L(A) \in \mathcal{D}$, we find $L(A) \subseteq L(B)$. Clearly, for all $j \in J$, we have $\bigcap_{j\in J} A_j \subseteq A_j$ and since L is monotonic, $L(\bigcap_{j\in J} A_j) \subseteq L(A_j)$ and this implies that $L(\bigcap_{j\in J} A_j) \subseteq \bigcap_{j\in J} L(A_j)$. By ($\mathbf{L}_3$), for all $j \in J$, we may write that $L(A_j) \subseteq A_j$ and then $\bigcap_{j\in J} L(A_j) \subseteq \bigcap_{j\in J} A_j$. However, $\mathcal{D}$ is a textural complete field, and so we have $\bigcap_{j\in J} L(A_j) \in \mathcal{D}$. This implies that $\bigcap_{j\in J} L(A_j) \subseteq L(\bigcap_{j\in J} A_j)$, that is, the proof of ($\mathbf{L}_2$) is complete. For ($\mathbf{L}_5$), note that

$$c_U(L(c_U(L(A)))) = H(L(A)) = \bigcap\{B \in \mathcal{U} \mid L(A) \subseteq B, B \in \mathcal{D}\} = L(A) \subseteq A$$

since $L(A) \in \mathcal{D}$. Since the operators L and H are dual, the dual conditions for the operator H can be proved by using a similar argument. Then by Theorem 5.4, there exists a complemented equivalence direlation (r, R) on $(U, \mathcal{U})$ where

$$L(A) = r^{\leftarrow}A, \text{ and } H(A) = R^{\leftarrow}A$$

for all $A \in \mathcal{U}$. Note that for $A \in \mathcal{D}$, we have $L(A) = A = H(A)$ and so clearly, we may write $\mathcal{D} = \{A \in \mathcal{U} \mid r^{\leftarrow}A = R^{\leftarrow}A\}$. □

8. Definability

Recall that for a given relation r on a universe U, a subset $X \subseteq U$ is *definable* if $\underline{apr}_r X = \overline{apr}_r X$ [16]. In this section, we show that definability can be stated in terms of inverse relation if the relation is reflexive and symmetric. We observe that if all sets are definable in a universe, then the relation is a function, and vice versa. Further, if it is given a complete field of sets $\mathcal{D}$, then we prove the existence of an equivalence relation such that $\mathcal{D}$ can be stated using definable sets. Essentially, the following theorems are natural results of textural discussions in the preceeding section. However, the proofs may be given independently.

Theorem 8.1. *Let r be a reflexive and symmetric relation on U and $A \subseteq U$. Then the following conditions are equivalent.*

(i) *A is definable.*

(ii) $U\backslash r^{-1}(A) = r^{-1}(U\backslash A)$.

(iii) $r^{-1}(A) = A$.

Proof. (i) $\Longrightarrow$ (ii) : Since A is definable, $\underline{apr}_r X = \overline{apr}_r X$, and by Theorem 5.5, we have

$$U\backslash r^{-1}(U\backslash A) = r^{-1}(A),$$

that is, $r^{-1}(U\backslash A) = U\backslash r^{-1}(A)$.

(ii) $\Longrightarrow$ (iii) : Since r is reflexive, clearly, $A \subseteq r^{-1}(A)$. Suppose that $r^{-1}(A) \nsubseteq A$. Let us choose a point $u \in U$ where $u \in r^{-1}(A)$ and $u \notin A$. Then by the assumption, we have $u \notin r^{-1}(U\backslash A)$ and this means that

$$v \in U\backslash A \implies (u, v) \notin r.$$

But $u \in U\backslash A$, and so we find that $(u, u) \notin r$. This implies a contradiction because of reflexivity of r.

(iii) $\Longrightarrow$ (i) : Since r is reflexive, we have $\underline{apr}_r A \subseteq \overline{apr}_r A$, that is

$$U\backslash r^{-1}(U\backslash A) \subseteq r^{-1}(A) = A.$$

Now we show that $A \subseteq U \backslash r^{-1}(U \backslash A)$. Suppose that $A \nsubseteq U \backslash r^{-1}(U \backslash A)$. Choose $a \in U$ such that $a \in A$ and $a \notin U \backslash r^{-1}(U \backslash A)$. Then $a \in r^{-1}(U \backslash A)$ and so for some $b \in U \backslash A$, we have $(a, b) \in r$. By symmetry, $(b, a) \in r$ and hence $b \in r^{-1}(\{a\}) \subseteq r^{-1}(A) = A$ which is a contradiction. $\square$

Let (r, R) be a difunction on $(U, \mathcal{P}(U))$. Then it is easy to see that DF1 and DF2 correspond to the following conditions, respectively, that is, r is an ordinary function on U:

(F1) For any $u \in U$ we have $w \in U$ such that $(u, w) \in r$.

(F2) If for some $w \in U$, $(w, u) \in r$ and $(w, v) \in r$ where $u, v \in U$, then $u = v$.

Theorem 8.2. *Let r be a relation on U. Then the following conditions are equivalent.*

(i) *r is a function on U.*

(ii) *$\forall X \subseteq U$, X is definable.*

(iii) *$\forall x \in U$, $\{x\}$ is definable.*

Proof. (i) $\Longrightarrow$ (ii) : Let r be a function on U and suppose that $\underline{apr}_r X \neq \overline{apr}_r X$. Then for some $x \in U$, we may write that $x \in U \backslash r^{-1}(U \backslash X)$ and $x \notin r^{-1}(X)$, and so we have

$$x \notin r^{-1}(X) \cup r^{-1}(U \backslash X) = r^{-1}(X \cup (U \backslash X)) = r^{-1}(U).$$

However, this is a contradiction, since $r^{-1}(U) = U$. Now let $x \notin U \backslash r^{-1}(U \backslash X)$ and $x \in r^{-1}(X)$. Then $x \in r^{-1}(U \backslash X)$ and this implies that $(x, y), (x, z) \in r$ for some $y, z \in U$ where $y \neq z$. Since r is a function, we also obtain a contradiction.

(ii) $\Longrightarrow$ (iii) : Immediate.

(iii) $\Longrightarrow$ (i) Now suppose that for all $x \in U$, $\{x\}$ is definable. If r is not a function, then we have two cases:

(a) For some $x, y, z \in U$ we have $(x, y), (x, z) \in r$ where $y \neq z$, or

(b) For some $x \in U$, $x \notin r^{-1}(U)$.

Consider the case (a). Note that $x \neq y$ or $x \neq z$. Let $x \neq y$. By the assumption, we have $U \backslash r^{-1}(U \backslash \{z\}) = r^{-1}(\{z\})$. Since $(x, z) \in r$, then $x \in r^{-1}(\{z\})$. However, $z \neq y$ implies that $y \in U \backslash \{z\}$. Hence, we have $x \in r^{-1}(U \backslash \{z\})$ and this gives that $x \notin U \backslash r^{-1}(U \backslash \{z\})$. But this contradicts to the equality $U \backslash r^{-1}(U \backslash \{z\}) = r^{-1}(\{z\})$. If $x \neq z$, we obtain a similar contradiction.

Now take the case (b). Since $\{x\}$ is definable, we have $U \backslash r^{-1}(U \backslash \{x\}) = r^{-1}(\{x\})$. Further, by (b), $x \notin r^{-1}(\{x\})$ and $x \notin r^{-1}(U \backslash \{x\})$ and so $x \in U \backslash r^{-1}(U \backslash \{x\})$ is a contradiction. $\square$

Theorem 8.3. *Let $\mathcal{D}$ be a complete field on U. Then there exists an equivalence relation r on U such that*

$$\mathcal{D} = \{A \subseteq U \mid \underline{apr}_r A = \overline{apr}_r A\}.$$

Furthermore,

$$[u]_r = \bigcap \{A \subseteq U \mid u \in A \text{ and } A \in \mathcal{D}\}$$

for all $u \in U$ where $[u]_r$ is the equivalence class with respect to r.

Proof. Consider the operators $L, H : \mathcal{P}(U) \to \mathcal{P}(U)$ defined by

$L(A) = \bigcup \{B \mid B \subseteq A \text{ and } B \in \mathcal{D}\}$

and

$H(A) = \bigcap \{C \mid A \subseteq C \text{ and } C \in \mathcal{D}\}$

for all $A \subseteq U$, respectively. We may use a similar argument as in the proof of Theorem 7.7 that the operators L and H satisfy the conditions of Theorem 5 in [16]. Hence, we have an equivalence relation r such that the operators L and H satisfy the equalities

$$L(A) = \underline{apr}_r A, \text{ and } H(A) = \overline{apr}_r A$$

where

$$\mathcal{D} = \{A \subseteq U \mid \underline{apr}_r A = \overline{apr}_r A\}.$$

Since $u \in [u]_r$ and $[u]_r \in \mathcal{D}$, clearly $\bigcap\{A \subseteq U \mid u \in A \text{ and } A \in \mathcal{D}\} \subseteq [u]_r$. Suppose that $[u]_r \not\subseteq \bigcap\{A \subseteq U \mid u \in A \text{ and } A \in \mathcal{D}\}$. Let us choose a point $v \in U$ where $v \in [u]_r$ and $v \notin \bigcap\{A \subseteq U \mid u \in A \text{ and } A \in \mathcal{D}\}$. This implies that $(u, v) \in r$ and for some $A \in \mathcal{D}$, we have $u \in A$ and $v \notin A$. Then $u \notin \underline{apr}_r A$ and so $u \notin L(A)$. But since $A \in \mathcal{D}$, $L(A) = H(A)$ and so $u \notin H(A)$. Now the inclusion $A \subseteq H(A)$ gives that $u \notin A$ which leads to a contradiction. □

The above result can be proved using textural concepts. To see this let us consider ordinary relations, and direlations in the context of discrete textures. For details on more general results, we refer to [14].

Let (r, R) be a direlation on the discrete texture $(U, \mathcal{P}(U))$. Then we have

$$r, R \in \mathcal{P}(U) \otimes \mathcal{P}(U) = \mathcal{P}(U) \times \mathcal{P}(U) = \mathcal{P}(U \times U),$$

that is, $r, R \subseteq U \times U$. Further,

$$\begin{aligned} r' &= \bigcap\{\overline{Q}_{(u,v)} \mid \exists w, z \in U \text{ with } r \not\subseteq \overline{Q}_{(w,z)}, c_U(Q_u) \not\subseteq Q_w \text{ and } P_z \not\subseteq c_U(P_v)\} \\ &= \bigcap\{(U \times U)\backslash\{(u, v)\} \mid \exists w, z \text{ with } (w, z) \in r, \{u\} \not\subseteq U\backslash\{w\} \text{ and } \{z\} \not\subseteq V\backslash\{v\}\} \\ &= (U \times U)\backslash \bigcup\{(u, v) \mid \exists w, z \text{ with } (w, z) \in r, u = w, v = z\} \\ &= (U \times U)\backslash \bigcup\{(u, v) \mid (u, v) \in r\} \\ &= (U \times U)\backslash r \end{aligned}$$

and

$$\begin{aligned} R' &= \bigvee\{\overline{P}_{(u,v)} \mid \exists w, z \text{ with } \overline{P}_{(w,z)} \not\subseteq R, P_w \not\subseteq c_U(P_u) \text{ and } c_U(Q_v) \not\subseteq Q_z\} \\ &= \bigcup\{\{(u, v)\} \mid \exists w, z \text{ with } (w, z) \notin R, w \notin U\backslash\{u\} \text{ and } v \notin U\backslash\{z\}\} \\ &= \bigcup\{(u, v) \mid \exists w, z \text{ with } (w, z) \notin R, u = w \text{ and } v = z\} \\ &= (U \times U)\backslash \bigcup\{(u, v) \mid (u, v) \in R\} \\ &= (U \times U)\backslash R. \end{aligned}$$

If (r, R) is a complemented direlation on $(U, \mathcal{U})$, then we have $R = r' = (U \times U)\backslash r$. Furthermore, the textural inverses of the relation and corelation are:

$$\begin{aligned} R^{\leftarrow} &= \bigvee\{\overline{P}_{(v,u)} \mid \overline{P}_{u,v)} \not\subseteq R\} \\ &= \{\{(v, u)\} \mid (u, v) \notin R\} = R^{-1} \end{aligned}$$

and

$$\begin{aligned} r^{\leftarrow} &= \bigcap\{\overline{Q}_{(v,u)} \mid r \not\subseteq \overline{Q}_{(u,v)}\} \\ &= \bigcap\{(U \times U)\backslash\{(v, u)\} \mid (u, v) \in r\} \\ &= (U \times U)\backslash\{\{(v, u)\} \mid (u, v) \in r\} = (U \times U)\backslash r^{-1}. \end{aligned}$$

It is easy to check that if (i_U, I_U) is the identity direlation on $(U, \mathcal{P}(U))$, then

$$i = \Delta = \{(u, u) \mid u \in U\} \text{ and } I = (U \times U)\backslash \Delta.$$

In view of Lemma 3.1 in [14], we have the following equivalences for the complemented direlations on discrete textures:

Theorem 8.4. *Let (r, R) be a direlation on $(U, \mathcal{P}U)$. Then*

(i) (r, R) *is a reflexive direlation on* $(U, \mathcal{P}(U)) \Longleftrightarrow r$ *is a reflexive relation on* U.

(ii) (r, R) *is a symmetric direlation on* $(U, \mathcal{P}(U)) \Longleftrightarrow r$ *is a symmetric relation on* U.

(iii) (r, R) *is a transitive direlation on* $(U, \mathcal{P}(U)) \Longleftrightarrow r$ *is a transitive relation on* U.

Proof. (i) it is immediate since $i \subseteq r \Longleftrightarrow \{(u, u) \mid u \in U\} \subseteq r$.
(ii) $r^{\leftarrow} = R \Longleftrightarrow (U \times U)\backslash r^{-1} = (U \times U)\backslash r \Longleftrightarrow r = r^{-1}$.
(iii) Since

$$\begin{aligned} r \circ r &= \bigvee\{\overline{P}_{(u,v)} \mid \exists w \in U \text{ with } r \not\subseteq \overline{Q}_{(u,w)} \text{ and } r \not\subseteq \overline{Q}_{(w,v)}\} \\ &= \{(u, v) \mid \exists w \in U \text{ with } (u, w) \in r \text{ and } (w, v) \in r\}, \end{aligned}$$

the textural composition is the usual composition of relations. This follows that r is a transitive relation on U. □

If $\mathcal{D}$ is a complete field of sets on U, then it is also a textural complete field of sets on the discrete texture $(U, \mathcal{P}(U))$. Then by Theorem 7.7, there exists a complemented equivalence direlation (r, R) on $\mathcal{P}(U)$ such that $\mathcal{D} = \{A \in \mathcal{P}(U) \mid r^{\leftarrow}A = R^{\leftarrow}A\}$. On the other hand, by Theorem 8.4, r is an equivalence relation on U. Further, by Theorem 5.5 we have

$$\forall X \subseteq U, \quad \underline{apr}_r X = r^{\leftarrow}X \quad \text{and} \quad \overline{apr}_r X = R^{\leftarrow}X.$$

This shows that the first part of Theorem 8.3 is a natural result of Theorem 7.7.

9. Complete Fields of Fuzzy Sets

Now we present some basic results on various fuzzy rough set algebras in the sense of Yao [16]. As it is mentioned in Sections 3 and 6, a fuzzy lattice (Hutton algebra) $\mathcal{F}(U)$ corresponds to a texture $(W_U, \mathcal{W}_U)$ which is called a Hutton texture, and a direlation between any two Hutton textures corresponds to a textural fuzzy direlation. This provides an easy way to observe the properties of definable fuzzy sets and complete fields of fuzzy sets on a fuzzy lattice $\mathcal{F}(U)$.

Definition 9.1. A subset $\mathcal{F} \subseteq \mathcal{F}(U)$ is called *a complete field of fuzzy sets* on $\mathcal{F}(U)$ if the following conditions hold:

(i) $\mathbf{1} \in \mathcal{F}$ where $\mathbf{1} : U \to [0, 1]$ is the function defined by $\forall u \in U$, $\mathbf{1}(u) = 1$.

(ii) $\mathcal{G} \subseteq \mathcal{F} \implies \bigvee \mathcal{G} \in \mathcal{F}$.

(iii) $\forall \alpha \in \mathcal{F}$, $\mathbf{1} - \alpha \in \mathcal{F}$.

Theorem 9.2. *If $\mathcal{F}$ is a complete field of fuzzy sets on $\mathcal{F}(U)$, then the family*

$$\mathcal{D} = \{\widehat{\alpha} \mid \alpha \in \mathcal{F}\}$$

is a textural complete field of sets on the corresponding texture $(W_U, \mathcal{W}_U)$.

Proof. Recall that the mapping $\widehat{\ }: \mathcal{F}(U) \to \mathcal{M}_{\mathcal{F}(U)}$ is a lattice isomorphism where $M_{\mathcal{F}(U)}$ is the set of all fuzzy points in $\mathcal{F}(U)$. Therefore the family $\mathcal{D}$ satisfies the following conditions:

(a) $\widehat{\mathbf{1}} = U \times (0, 1] = W_U \in \mathcal{D}$.

(b) Let $\{\widehat{\alpha_j} \mid j \in J\} \subseteq \mathcal{D}$. Then $\{\alpha_j \mid j \in J\} \subseteq \mathcal{F}$. Since $\mathcal{F}$ is a complete field of fuzzy sets in $\mathcal{F}(U)$, we have $\bigvee\{\alpha_j \mid j \in J\} \in \mathcal{F}$ and so we may write that

$$\bigvee_{j \in J} \widehat{\alpha_j} = \widehat{\bigvee_{j \in J} \alpha_j} \in \mathcal{D}.$$

(c) For all $\alpha \in \mathcal{F}$, we have $\mathbf{1} - \alpha \in \mathcal{F}$ and so $\widehat{\mathbf{1} - \alpha} \in \mathcal{D}$. By definition of the complementation c of $(W_U, \mathcal{W}_U)$, we have

$$\widehat{\mathbf{1} - \alpha} = c(\widehat{\alpha}) \in \mathcal{D}.$$

Hence, $\mathcal{D}$ is a textural complete field of sets on $(W_U, \mathcal{W}_U)$. $\square$

Theorem 9.3. *Let (ϕ, Φ) be a reflexive textural fuzzy direlation on $\mathcal{F}(U)$. Then the family*

$$\mathcal{F} = \{\alpha \in \mathcal{F}(U) \mid \phi^{\leftarrow}\alpha = \Phi^{\leftarrow}\alpha\}$$

is a complete field of fuzzy sets on $\mathcal{F}(U)$.

Proof. By Theorem 2.3 in [15], there is a direlation (r, R) on $(W_U, \mathcal{W}_U)$ where $\mu_r = \phi$ and $\mu_R = \Phi$. By Theorem 7.1 (i) in [5], (r, R) is also a reflexive direlation on $(W_U, \mathcal{W}_U)$. Then by Theorem 7.6 (i), the family

$$\mathcal{D} = \{\widehat{\alpha} \mid r^{\leftarrow}\widehat{\alpha} = R^{\leftarrow}\widehat{\alpha},\ \alpha \in \mathcal{F}(U)\}$$

is a textural complete field of sets on $(W_U, \mathcal{W}_U)$. On the other hand, by definition of presections of direlations and textural fuzzy direlations, we have

$$r^{\leftarrow}\widehat{\alpha} = \widehat{\beta} \iff \phi^{\leftarrow}\alpha = \beta,$$

$$R^{\leftarrow}\widehat{\alpha} = \widehat{\gamma} \iff \Phi^{\leftarrow}\alpha = \gamma.$$

Since the lattice isomorphism $\widehat{\ }$ is injective, $r^{\leftarrow}\widehat{\alpha} = R^{\leftarrow}\widehat{\alpha}$ implies that $\beta = \gamma$. Hence, we find $\phi^{\leftarrow}\alpha = \Phi^{\leftarrow}\alpha$. Essentially, we have

$$r^{\leftarrow}\widehat{\alpha} = R^{\leftarrow}\widehat{\alpha} \iff \phi^{\leftarrow}\alpha = \Phi^{\leftarrow}\alpha$$

for all $\alpha \in \mathcal{F}(U)$. We have concluded that the family $\mathcal{F}$ satisfies the desired conditions. $\square$

Theorem 9.4. *Let $\mathcal{F}$ be a complete field of fuzzy sets on $\mathcal{F}(U)$. Then there exists an equivalence textural fuzzy direlation (ϕ, Φ) such that*

$$\mathcal{F} = \{\alpha \mid \phi^{\leftarrow}\alpha = \Phi^{\leftarrow}\alpha\}.$$

Proof. By Theorem 9.2, $\mathcal{D} = \{\widehat{\alpha} \mid \alpha \in \mathcal{F}\}$ is a textural complete field of sets on $(W_U, \mathcal{W}_U)$. Then by Theorem 7.7, there is an equivalence direlation (r, R) on $(W_U, \mathcal{W}_U)$ such that $\mathcal{D} = \{\widehat{\alpha} \mid r^{\leftarrow}\widehat{\alpha} = R^{\leftarrow}\widehat{\alpha}\}$. Take the corresponding textural fuzzy direlation (ϕ, Φ) on $\mathcal{F}(U)$ where $\phi = \mu_r$ and $\Phi = \mu_R$. By Theorem 7.1 in [5], (ϕ, Φ) is an equivalence textural fuzzy direlation on $\mathcal{F}(U)$. Further, we have

$$\mathcal{F} = \{\alpha \in \mathcal{F}(U) \mid \widehat{\alpha} \in \mathcal{D}\} = \{\alpha \in \mathcal{F}(U) \mid r^{\leftarrow}\widehat{\alpha} = R^{\leftarrow}\widehat{\alpha}\} = \{\alpha \in \mathcal{F}(U) \mid \phi^{\leftarrow}\alpha = \Phi^{\leftarrow}\alpha\}. \quad \square$$

Now let φ be a fuzzy relation, that is, $\varphi \in \mathcal{F}(U \times U)$. Recall that the pair $(\phi_\varphi, \Phi_\varphi)$ is a textural fuzzy direlation on $\mathcal{F}(U)$ where

$$\phi_\varphi(u, s, v) = \varphi(u, v) \wedge s \text{ and } \Phi_\varphi(u, s, v) = (1 - \varphi(u, v)) \vee s$$

for all $(u, s, v) \in U \times [0, 1] \times U$ [15]. For any $\alpha \in \mathcal{F}(U)$, the presections are the fuzzy sets, that is,

$$\forall u \in U,\ \Phi_\varphi^{\leftarrow}\alpha(u) = \overline{apr}\alpha(u) = \bigwedge\{s \in [0, 1] \mid \alpha(v) \le 1 - \varphi(u, v) \vee s,\ v \in U\}$$

and

$$\phi_\varphi^{\leftarrow}\alpha(u) = \underline{apr}\alpha(u) = \bigvee\{s \in [0, 1] \mid \varphi(u, v) \wedge s \le \alpha(v),\ v \in U\}.$$

Further, the system $(\mathcal{F}(U), \wedge, \vee, \sim, \Phi_\varphi^{\leftarrow}, \phi_\varphi^{\leftarrow})$ is a fuzzy rough set algebra [5].

Then we may write the following result as a corollary of Theorem 9.3:

Corollary 9.5. *Let φ be a reflexive fuzzy relation. Then the family*

$$\mathcal{F} = \left\{\alpha \in \mathcal{F}(U) \mid \Phi_\varphi^{\leftarrow}\alpha = \phi_\varphi^{\leftarrow}\alpha\right\}$$

is a complete field of fuzzy sets on $\mathcal{F}(U)$.

For a fuzzy relation $\varphi : U \times U \to [0, 1]$, the system

$$\left(\mathcal{F}(U), \wedge, \vee, \sim, \phi_\varphi^{\rightarrow}, \Phi_\varphi^{\rightarrow}\right)$$

is also a fuzzy rough set algebra and this gives the fuzzy rough sets

$$\begin{aligned} \forall u \in U,\ \Phi_\varphi^{\rightarrow}\alpha(u) &= \underline{apr}\alpha(u) = \bigwedge\{1 - \varphi(v, u) \vee \alpha(v) \mid v \in U\}, \\ \phi_\varphi^{\rightarrow}\alpha(u) &= \overline{apr}\alpha(u) = \bigvee\{\varphi(v, u) \wedge \alpha(v) \mid v \in U\} \end{aligned}$$

of Pei's model under the assumption that φ has the symmetry property (see [13] and Example 2.26 in [15]).

Finally, using dual arguments we have the following.

Corollary 9.6. *Let φ be a reflexive fuzzy relation. Then the family*

$$\mathcal{F} = \{\alpha \in \mathcal{F}(U) \mid \Phi_\varphi^{\rightarrow}\alpha = \phi_\varphi^{\rightarrow}\alpha\}$$

is a complete field of fuzzy sets on $\mathcal{F}(U)$.

10. Conclusion

Direlations are suitable morphisms for textures and they play an important role in texture space theory (see e.g., [3,7]). Presections and sections with respect to direlations provide a different perspective for the basic properties of rough sets. Hence, they are also essential tools for an abstract model of rough set theory [7]. It is known that textures are alternative point-set based setting for fuzzy sets. In this setting, direlations between Hutton textures turn into the textural fuzzy direlations between fuzzy lattices [15]. In view of this fact, using definability we have observed that the complete field of fuzzy sets can be stated for the various fuzzy rough set algebras which are studied in the literature of rough set theory. On the way, we have given a partial affirmative answer to an open problem related to fuzzy approximation spaces imposed by Wu in [20].

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